Linking a Minimal Theoretical Model of Scale Factor Evolution with Observational Universe Information

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Abstract

We study several approaches for constructing a minimal model of Universe evolution by matching different stages of scale factor laws. We discuss the continuity in the transitions among the stages and the time variables involved. We develop a way to modelize these transitions without loss of observational predictability. The key is the scale factor and its connection with minimal but well stablished observational information (transition times and expansion ratii). This construction is clearly useful in metric perturbations type computations, but it can be applied in whatever subject dealing with a cosmology involving different evolution stages.

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1 Introduction

Cosmology has a rich variety of theories giving rise to different evolution laws for the Universe. Realistic models for the Universe must attach with the available observational data. Among them, the age of Universe and some known transition times must be considered.

From the theoretical point of view, the usual process consists in matching successives stages having different evolution laws each one. Particular interest is devoted to works computing the generation of metric perturbations, since they must handle the transitions among different evolution stages. Some minimal conditions of continuity (of scale factor and the perturbations) are required.

Different ways for handling these continuity conditions can be found ([1]-[7]). Some of them uses impose continuity on the cosmic time evolution although discontinuity in conformal time itself ([2],[3]), while others make a continuous matching of the scale factor in conformal time variable ([1],[4],[5],[6],[7]). Both cases would have an imprint in the successive study of the process of interest like metric perturbations propagation, since of the parameters involved in the linking of the different stages from the very early to the current one. At time of confronting with observations, the parameters left by the linking process can play a crucial role.

The situation is still confuse when time scales must be fixed on those minimal models. Some kind of scale factor normalizations ([2]) or energy scales at transitions ([7]) are usually imposed in order to overcome the loss of predictability of the temporal variables used.

Our scope is to review this process in constructing the evolution of these minimal Universe models. Lacking a complete underlying theory describing the total Universe evolution or at least, the detail of transitions between stages, one must join these stages in order to compose an *asymptotically* suitable description. The way how handle each stage and how make the matching among them is the subject of this paper. We try to apply a different and more practical treatment which preserves the level of *predictability* of the minimal models discussed.

From the theoretical point of view, different evolution laws can be provided for the Universe: inflationary expansion of power law type, De Sitter type, inverse power type, deccelerated and standard expansion stages. From the observational side, the Universe appears to be homogeneus and isotropic at sufficiently large scales, following the distribution of clusters of galaxies. From the measurements of redshift in the spectra of galaxies, the Universe has a metric in expansion. The measurements of current Hubble factor give us an order of the age of Universe of $\mathcal{T}_0 \sim 10^{17}s$, at least as upper limit. We find ourselves in an Universe with an expansion rate matter dominated-type . We know too the previous existence of a hotter and denser radiation dominated expansion, as can be assured from the current existence of the Cosmic Microwave Background Radiation. The end of radiation dominated stage is estimated at $\mathcal{T}_m \sim 10^{12}s$.

Finally, a mixed theoretical-observational interpretation had led to suppose the existence of an inflationary stage in the very early beginning. This is the so called inflationary paradigm, whose exponential version looks able to solve the cosmological problems on closure and flatness of Universe, large scale homogeneity and baryon number. (See for example [8] and [9]).

To explain the whole Universe evolution would mean to explain phenomena at very different energy scales. At the date, we lack of such a complete theory. Therefore, the tools of theoretical cosmology are theories with their own ranges of validity. The ideal total description is substituted by a description made with successive stages, giving a step-by-step evolution. Thus, we find the need of joining the different and in principle discontinuous evolution stages in order to make the most approximative description.

It leds us to deal with modelized transitions, *not necessarily corresponding to real transitions*. In the minimal models, these modelized transitions are very limited with respect to describe and account the real transitions. The logical choice of a continuous time variable is yet not trivial and different options are possible. Further computations as gravitational waves and imprints on CMB are signed by these transitions and can be signed by these choices.

We study this problem by applying a more practical point of view: closer to the observational Universe, better description. This guideline has suggested us an appropiate treatment of the temporal variables and transitions involved.

What do we mean by a minimal model? An *asymptotically* valuable description of the scale factor evolution running on a temporal variable. Real transitions among the different stages, probably discontinuous, are modelized trough this mathematical temporal-type variable in a more convenient way. Further linking with observational Universe information preserves the prediction capability of the descriptive minimal model thus constructed. Every theoretical cosmology requires as its basic tool a scale factor and a temporal variable over which runs. The minimal observational Universe information is translated on the theoretical model evolution through such scale factor function $a(t)$ and its transitions ([10]). Suitable temporal variables for this process of translation are not a trivial subject. A compromise among the theoretical computational ability and the observational Universe information contained is required.

We find the particular proper times in each stage not to be appropiated variables in the matching process, because their inherent discontinuity in a minimal model. Computations make desiderable to have a minimal conditions of continuity of scale factor and temporal variable. One can ask if real transitions are continuous at the level of such minimal descriptions. The answer is probably "not". Now, it makes necessary to reinterpretate the meaning and role of the temporal variable involved in this minimal model descriptions and distinguish it from the physical proper time.

Need of deal with different temporal variables and the transformations relating them can be seen at least as the prize to pay for our lack of knowledge about the dynamics of real transitions among the cosmological different evolution stages. With this lacking of knowledge it would be impossible to construct a continuous, full model for the scale factor including the transitions. We overcome this problem by defining auxiliary *descriptive* temporal variables. These variables make a closer mapping of the proper cosmic time. The freedom in their boundary conditions can be used in order to make them satisfactorily continuous through transitions. With the use of such an intermediate tool we must rewrite the evolution laws in these variables.

A careful handling allows us to maintain a link with the observational temporal parameters and thus, abilitates our construction to be worked out in order to further extraction of other observational cosmological phenomena. The minimal observational Universe information introduced are the time scales, by taking the well stablished standard values for beginning of radiation dominated stage ($\mathcal{T}_r \sim 10^{-32}s$), the beginning of matter dominated stage ($\mathcal{T}_m \sim 10^{12}s$) and the proper scale factor expansion ratii to be reached in each one of the stages.

In the next, we develop the descriptive cosmic time-type variables for minimal models with an inverse power inflationary stage. We have chosen this kind of inflationary behaviours since they appear typically in the context of String Cosmology. The simple guidelines here showed could be easily followed with whatever other inflationary type behaviour. The scale factor description obtained reaches the proper scale factor ratii in each stage. The transitions among different stages are modelized as sudden, continuous and smooth. All parameters and variables in the description have been linked with observational transition times and proper cosmic time evolution.

We analyze also the conformal time variable and the corresponding scale factor. This construction is made over descriptive temporal variables. We elaborate transformation rules in order to keep the link among the conformal time variable and observational times. Conditions of continuity and simplicity can be imposed on the most general construction of conformal time variables. The scale factor evolution thus obtained has sudden, continuous and smooth transitions both in the cosmic time description and in the conformal time one. This feature of our treatement is not an usual property for most cosmological minimal models.

We elaborate also another conformal time construction. It is linked to observational information and proper cosmic times in a more direct way. The scale factor in this variable can be written with sudden and instantaneous transitions, but they are not smooth.

In appendixes we report some other related aspects. We apply the descriptive variables treatment to a minimal model with arbitrary power law inflationary stage. The possible modelization of transitions as brief intermediate stages without descriptive variables is considered also. Finally, an alternative construction of descriptive variables is included.

2 The Temporal Coordinate

Solutions describing an inflationary stage (I), radiation dominated stage (II) and matter dominated stage (III) can be obtained from different early Universe theories providing cosmological backgrounds. We consider them extracted in the proper cosmic time gauge, where the metric is written as:

$$ds^2 = dt^2 - a(t)^2 dx^2$$

$a(t)$ is the scale factor in cosmic time variable, giving the expansion rate of the metric. Notice here $t = c\mathcal{T}$. (In natural units, where $\hbar = G = c = 1$, the equality among variables t and $\mathcal{T}$ holds).

In order to construct a minimal model that describes the evolution of the scale factor, we will consider these successive three stages in descriptive cosmic time type variables as having instantaneous and continuous transitions at times $\bar{t}_1$ and $\bar{t}_2$. With full free parameters, we write our model as:

$$\begin{array}{rcl} \bar{\bar{a}}_I(\bar{\bar{t}}) = \bar{\bar{A}}_I(\bar{\bar{t}}_I - \bar{\bar{t}})^{-Q} & & \bar{\bar{t}} < \bar{t}_1 \\ \bar{a}_{II}(\bar{t}) = \bar{A}_{II}(\bar{t} - \bar{t}_{II})^{R} & & \bar{t}_1 < \bar{t} < \bar{t}_2 \\ a_{III}(t) = A_{III}(t - t_{III})^{M} & & \bar{t}_2 < t \end{array} \tag{2.1}$$

Here the first stage (I) describes an accelerated expansion provided $\bar{\bar{t}}_I > \bar{t}_1$ and $Q > 0$. This is the inverse power inflationary stage. In appendixes we consider the case for arbitrary power law inflationary stages. The behaviours of radiation and matter dominated stages are identified by power law dependences with exponents $R = \frac{1}{2}$ and $M = \frac{2}{3}$ in cosmic time gauge. By imposing the continuous matching of the scale factor and its first derivative at the transition times $\bar{t}_1$ and $\bar{t}_2$, give us four constraints on the six free parameters. We have choose to express the parameters $\bar{\bar{t}}_I$, $\bar{\bar{A}}_I$, t_{III}, A_{III} in function of $\bar{t}_{II}$, $\bar{A}_{II}$. We have:

$$\begin{aligned}
\bar{\bar{t}}_I &= \bar{t}_1 \left(1 + \frac{Q}{R}\right) - \bar{t}_{II}\frac{Q}{R} \\
\bar{\bar{A}}_I &= \bar{A}_{II}\left(\frac{Q}{R}\right)^Q (\bar{t}_1 - \bar{t}_{II})^{R+Q} \\
t_{III} &= \left(1 - \frac{M}{R}\right)\bar{t}_2 + \frac{M}{R}\bar{t}_{II} \\
A_{III} &= \bar{A}_{II}\left(\frac{R}{M}\right)^M (\bar{t}_2 - \bar{t}_{II})^{R-M}
\end{aligned} \tag{2.2}$$

The inflationary stage is described with a time variable $\bar{\bar{t}}$ and the radiation stage with a variable $\bar{t}$. Notice that these cosmic time-type variables (and t of the matter dominated stage) are not at priori exactly equal to the physical proper time coordinate (multiplied by c in no natural units), but transformations (dilatation plus translation) of it.

We will find relations between these variables and the physical time coordinate (multiplied by c, $t = c\mathcal{T}$) which allow to put in contact this description with the minimal and well stablished observational Universe information: the scale times (standard transition times) and the proper scale factor expansion ratii. We understand that a descriptive temporal variable can provide a better scale factor description. The cosmic time type variables here introduced act as intermediate vehicles in the process of linking observational Universe and theoretical evolution model. The minimal information about the former one is imprinted on transformations giving raise such descriptive variables, while remaining parameters can be fixed in order to have a suitable scale factor continuity for cosmology purposes.

3 Link with the Observational Universe

3.1 The Observational Universe

We consider now some properties of the observational Universe evolution. Taking the simplest option, we consider the "real" scale factor evolution to have transitions at t_r (beginning of radiation dominated stage) and t_m (beginning of matter dominated stage), given by the standard observational values. We define a beginning of inflation at t_i and corresponding current time t_0. Equality in the temporal dependence exponents in the real and descriptive scale factors is preserved, the other free parameters are related by transitions.

In order to have the right temporal dependence (following the real duration and scale factor expansion of the observational Universe) for a radiation dominated behaviour and a matter dominated behaviour, we must consider there a scale factor without additive constants in their proper cosmic time dependences. Notice as a consequence, we can not consider continuity in first derivative of scale factor in the "real" transitions. Continuity on the scale factor itself is supposed by considering real transitions enoughly briefs and smooths for not requiring be described as intermediate stages.

$$
\begin{aligned}
a_I(t) &= A_I(t_I - t)^{-Q} \qquad t \in (t_i, t_r) \\
a_{II}(t) &= A_{II}\, t^R \qquad\qquad t \in (t_r, t_m) \\
a_{III}(t) &= A_{III}\, t^M \qquad\qquad t \in (t_m, t_0)
\end{aligned}
\tag{3.1}
$$

Without knowing the dynamics of the real transitions among stages, this approach still allow us to extract a robust constraint from the observational knowledge of our Universe: the scale factor expansion (or scale factor ratii) reached in each one of the three stages considered. Meanwhile the duration of a stage, the scale factor will expand, in the inflationary stage:

$$
\frac{a_I(t_r)}{a_I(t_i)} = \left(\frac{t_I - t_r}{t_I - t_i}\right)^{-Q} \qquad , \tag{3.2}
$$

in the radiation dominated stage:

$$
\frac{a_{II}(t_m)}{a_{II}(t_r)} = \left(\frac{t_m}{t_r}\right)^{R} \qquad , \tag{3.3}
$$

and in the matter dominated stage:

$$
\frac{a_{III}(t_0)}{a_{III}(t_m)} = \left(\frac{t_0}{t_m}\right)^{M} \tag{3.4}
$$

We consider the standard observational values for cosmological times. The radiation-matter transition held at:

$$
\mathcal{T}_m \sim 10^{12} s \quad ,
$$

the beginning of radiation stage at

$$
\mathcal{T}_r \sim 10^{-32} s \quad ,
$$

and the current age of the Universe

$$
\mathcal{T}_0 \sim {H_0}^{-1} \sim 10^{17} s \qquad .
$$

The exact numerical value of $\mathcal{T}_0$ turns out not be crucial here.

3.2 Linking Model and Observations

We can impose now to our description to satisfy stage by stage the same right scale factor ratii as the observable evolution. For the matter dominated stage, it is convenient to have its descriptive scale factor written in proper cosmic time, since it is within this stage where observational information must be extracted and provided. Then, we fix one of the two still free parameters in matching eqs.(2.2) by eliminating in eq.(2.1) the additive constant t_{III}:

$$t_{III} \equiv 0 \tag{3.5}$$

Now, in order to have the right scale factor ratio during the whole matter dominated stage, the beginning $\bar{t_2}$ must coincide exactly with the beginning of matter dominated stage t_m.

$$\bar{t_2} \equiv t_m \tag{3.6}$$

This first link with observation translates immediately on the radiation dominated stage description. From eq.(2.2) and with the obtained conditions eqs.(3.5) and (3.6), the additive constant t_{II} is fixed.

$$\bar{t_{II}} = \left(1 - \frac{R}{M}\right) t_m \tag{3.7}$$

In order to fix the beginning of radiation stage $\bar{t_1}$, we use the observational value of scale factor expansion during the whole stage eq.(3.3) and make it coincident with the value obtained in the descriptive scale factor model. By this way:

$$\frac{\bar{a_{II}}(\bar{t_2})}{\bar{a_{II}}(\bar{t_1})} = \frac{\bar{A_{II}}(\bar{t_2} - \bar{t_{II}})^R}{\bar{A_{II}}(\bar{t_1} - \bar{t_{II}})^R} = \left(\frac{t_m}{t_r}\right)^R$$

which gives the relation:

$$\bar{t_1} = t_r + \left(1 - \frac{t_r}{t_m}\right) \bar{t_{II}} \quad .$$

With the condition eq.(3.7), we obtain the expression of $\bar{t_1}$ as function of the observational transitions t_r and t_m:

$$\bar{t_1} = \frac{R}{M} t_r + \left(1 - \frac{R}{M}\right) t_m \tag{3.8}$$

We can interpolate this process for every time t belonging to the radiation dominated stage, and require equally to the model to give the corresponding scale factor expansion ratio at $\bar{t}$. Thus, it is possible to get a relation linking the temporal variable $\bar{t}$ with the physical time t:

$$\bar{t} = \frac{R}{M}t + \left(1 - \frac{R}{M}\right) t_m \tag{3.9}$$

The conditions on $\bar{t_1}$ have fixed yet the end of the descriptive inflationary stage. For fixing the corresponding beginning at $\bar{\bar{t_i}}$, the same above guidelines are followed. The total expansion during the inflationary stage (3.2) must be:

$$\frac{\bar{\bar{a}}_I(\bar{t_1})}{\bar{\bar{a}}_I(\bar{\bar{t_i}})} = \frac{\bar{\bar{A}}_I(\bar{t_1} - \bar{\bar{t}}_I)^{-Q}}{\bar{\bar{A}}_I(\bar{\bar{t_i}} - \bar{\bar{t}}_I)^{-Q}} = \left(\frac{t_r - t_I}{t_i - t_I}\right)^{-Q}$$

From this we obtain:

$$\bar{\bar{t_i}} = \frac{\bar{\bar{t}}_I\,(t_i - t_r) \;+\; \bar{t_1}\,(t_I - t_i)}{(t_I - t_r)}$$

and making use of relations (3.8), (3.6) and the matching equations (2.2)

$$\bar{\bar{t_i}} = \left(\frac{R}{M} + \frac{Q}{M}\frac{t_r - t_i}{t_r - t_I}\right) t_r + \left(1 - \frac{R}{M}\right) t_m \tag{3.10}$$

Again, we can interpolate this relation for all t in the inflationary stage and obtain the relation for the temporal variable $\bar{\bar{t}}$ in terms of the observational transition values t_r and t_m:

$$\bar{\bar{t}} = \left(\frac{R}{M} + \frac{Q}{M}\frac{t - t_i}{t_r - t_I}\right) t_r + \left(1 - \frac{R}{M}\right) t_m \tag{3.11}$$

3.3 Summary of Scale Factor Description

Equations (3.2-3.4) have allowed us to link the model for the scale factor eqs.(2.1-2.2) with observational Universe information. The scale factor written in cosmic time-type variables have thus the expression:

$$\begin{aligned}
\bar{\bar{a}}_I(\bar{\bar{t}}) &= \bar{\bar{A}}_I(\bar{\bar{t}}_I - \bar{\bar{t}})^{-Q} & \bar{\bar{t_i}} < \bar{\bar{t}} < \bar{t_1} \\
a_{\bar{II}}(\bar{t}) &= A_{\bar{II}}(\bar{t} - t_{\bar{II}})^{R} & \bar{t_1} < \bar{t} < \bar{t_2} \\
a_{III}(t) &= A_{III}(t)^{M} & \bar{t_2} < t < t_0
\end{aligned} \tag{3.12}$$

with sudden and continuous transitions at $\bar{t_1}$ and $\bar{t_2}$ for both the scale factor and first derivatives with respect to the descriptive cosmic time-type variables $\bar{\bar{t}}$ eq.(3.11), $\bar{t}$ eq.(3.9) and t. In such a way, the parameters in function of transitions and global scale factor $\bar{A}_{II}$:

$$\bar{\bar{t}}_I = \bar{t_1}\left(1 + \frac{Q}{R}\right) - \bar{t_2}\left(\frac{Q}{R} - \frac{Q}{M}\right) \quad , \quad t_{\bar{II}} = \left(1 - \frac{R}{M}\right)\bar{t_2} \tag{3.13}$$

$$\bar{\bar{A}}_I = A_{\bar{II}}\left(\frac{Q}{R}\right)^{Q}\left(\bar{t_1} - (1 - \frac{R}{M})\bar{t_2}\right)^{R+Q} \quad , \quad A_{III} = A_{\bar{II}}\left(\frac{R}{M}\right)^{R}\bar{t_2}^{\,R-M}$$

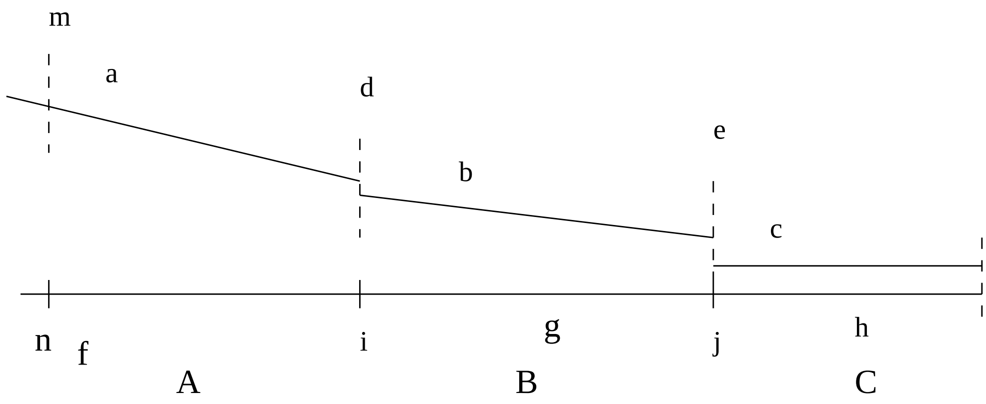

Figure 1: Linking with observational information by descriptive variables

Representation of the linking process with observational Universe information. The theoretical minimal model is found in the lower side, running on variables $\bar{\bar{t}}$, $\bar{t}$, t with sudden and continuous transitions on such variables at $\bar{t}_1$ and $\bar{t}_2$. The observational Universe information is represented in the upper side, whose dynamics at transitions t_r and t_m can be considered nearly sudden but in any case discontinuous which respect to proper cosmic times. Expressions relating both pieces are given in the text.

We remember that transitions $\bar{t}_1$, $\bar{t}_2$ and the beginning of the inflationary stage description $\bar{\bar{t}}_i$ are expressed in terms of the standard observational times t_r and t_m as:

$$\begin{aligned} \bar{t}_1 &= \frac{R}{M} t_r + \left(1 - \frac{R}{M}\right) t_m \qquad , \qquad \bar{t}_2 = t_m \\ \bar{\bar{t}}_i &= \left(\frac{R}{M} + \frac{Q}{M}\frac{t_r - t_i}{t_r - t_I}\right) t_r + \left(1 - \frac{R}{M}\right) t_m \end{aligned} \tag{3.14}$$

Parameters of scale factor (3.12) satisfy also the corresponding equations (2.2). With help of eqs.(3.5),(3.7) and (3.14), these parameters can be written too in terms of t_r, t_m and the global scale factor $\bar{A}_{II}$:

$$\begin{aligned} \bar{\bar{t}}_I &= t_r\left(\frac{R}{M} + \frac{Q}{M}\right) + t_m\left(1 - \frac{R}{M}\right) \quad , \quad \bar{t}_{II} = \left(1 - \frac{R}{M}\right) t_m \\ \bar{\bar{A}}_I &= \bar{A}_{II}\left(\frac{Q}{M}\right)^Q \left(\frac{R}{M}\right)^R {t_r}^{R+Q} \qquad , \quad A_{III} = \bar{A}_{II}\left(\frac{R}{M}\right)^R {t_m}^{R-M} \end{aligned} \tag{3.15}$$

The descriptive scale factor eq.(3.12) can be expressed also in the proper cosmic time variables as:

$$\begin{aligned} \bar{\bar{a}}_I(t) &= \bar{A}_{II}\left(\frac{R}{M}\right)^R {t_r}^{R+Q} \left(\frac{t_I}{t_r} - 1\right)^Q (t_I - t)^{-Q} \\ \bar{a}_{II}(t) &= \bar{A}_{II}\left(\frac{R}{M}\right)^R t^R \\ a_{III}(t) &= \bar{A}_{II}\left(\frac{R}{M}\right)^R {t_m}^{R-M}\, t^M \end{aligned} \tag{3.16}$$

but remember that not fully continuity (and not in first derivatives) helds on proper cosmic time. It could be considered sudden instantaneous transitions at t_r and t_m in such a way that the scale factor is continuous:

$$\bar{\bar{a}}_I(t_r) = \bar{a}_{II}(t_r)$$
$$\bar{a}_{II}(t_m) = a_{III}(t_m)$$

but there is not continuity in the first time derivative of the scale factor. In the inflation-radiation dominated transition t_r, this continuity condition would be satisfied by models whose inflation scale factor follows:

$$t_I = \left(1 + \frac{Q}{R}\right) t_r \tag{3.17}$$

and is in anycase impossible to obtain derivative continuity at the radiation dominated-matter dominated transition t_m. In sake of generality we will not consider eq.(3.17) holds in the following.

By the other hand, in the transformed descriptive variables, the transitions are much more smooth. In fact, we know that $\bar{\bar{a}}_I(\bar{t_1}) = \bar{a}_{II}(\bar{t_1})$ and $\bar{a}_{II}(\bar{t_2}) = a_{III}(\bar{t_2})$. It holds also:

$$\left.\frac{d\bar{\bar{a}}_I(\bar{\bar{t}})}{d\bar{\bar{t}}}\right|_{\bar{t_1}^-} = \left.\frac{d\bar{a}_{II}(\bar{t})}{d\bar{t}}\right|_{\bar{t_1}^+} \quad , \quad \left.\frac{d\bar{a}_{II}(\bar{t})}{d\bar{t}}\right|_{\bar{t_2}^-} = \left.\frac{da_{III}(t)}{dt}\right|_{\bar{t_2}^+}$$

Notice that:

$$\frac{d\bar{\bar{t}}}{dt} = -\frac{Q}{M}\frac{t_r}{t_r - t_I} \quad , \quad \frac{d\bar{t}}{dt} = \frac{R}{M} \tag{3.18}$$

These are the factors absorbing the discontinuity in the derivative with respect to proper cosmic time and allowing continuity in the first derivative of scale factor in descriptive variables $\bar{\bar{t}}$ and $\bar{t}$ at transitions $\bar{t_1}$ and $\bar{t_2}$.

The minimal model thus constructed reaches the same expansion ratii in each stage and with the same evolution laws than extracted from information about observational Universe. But differently from that, this model has sudden, continuous and smooth transitions among the three stages. A feature making possible this effect is a change in the duration of stages. When measured in descriptive variables, matter dominated stage has the same duration than in observational description. Not the same for radiation dominated and inflationary stages:

$$\Delta\bar{t}|_{RAD} = \frac{R}{M}\,\Delta t|_{RAD}$$
$$\Delta\bar{\bar{t}}\Big|_{INF} = \frac{Q}{M}\frac{t_r}{t_I - t_r}\,\Delta t|_{INF}$$

For usual values of R and M, the radiation dominated stage suffers a contraction in descriptive variables. For inflationary stage, it depends of involved parameters. Contraction takes place in this stage if $t_I > \left(\frac{Q}{M} + 1\right) t_r$.

4 The Conformal Time Coordinate

In the last Section we have obtained a minimal model for the scale factor, working with descriptive variables of cosmic time type. We have related those with the observational transition times. It is also convenient to translate it in terms of the conformal time. This variable is defined as $d\eta = \frac{dt}{a(t)}$. The metric in conformal time gauge is expressed as:

$$ds^2 = a(\eta)^2 \left(d\eta^2 - dx^2\right)$$

Following our definition of cosmic time type variable $t = c\mathcal{T}$, the variable η will have too dimensions of lenght.

4.1 The general case

We construct now the conformal time over the descriptive cosmic time variables, with which the scale factor is satisfactorily continuous. That means we will have a particular conformal time variable for each stage, not only oughted to the different shape of the scale factor but also as consequence of the different temporal variables used in its description. Thus, following eqs.(3.12) we define:

$$\begin{aligned} d\,\bar{\bar{\eta}} &= \frac{d\bar{\bar{t}}}{\bar{\bar{a}}_I(\bar{\bar{t}})} \qquad \bar{\bar{t}}_i < \bar{\bar{t}} < \bar{t}_1 \\ d\,\bar{\eta} &= \frac{d\bar{t}}{\bar{a}_{II}(\bar{t})} \qquad \bar{t}_1 < \bar{t} < \bar{t}_2 \\ d\,\eta &= \frac{dt}{a_{III}(t)} \qquad \bar{t}_2 < t < t_0 \end{aligned} \tag{4.1}$$

We integrate these equations in order to obtain expressions for the conformal time variable in each stage. Let be the parameters η_i, η_1 and η_2 the conformal time values corresponding to the beginning of each stage of the minimal model at $\bar{\bar{t}}_i$, $\bar{t}_1$ and $\bar{t}_2$.

$$\eta_i = \bar{\bar{\eta}}(\bar{\bar{t}}_i) \quad , \quad \eta_1 = \bar{\eta}(\bar{t}_1) \quad , \quad \eta_2 = \eta(\bar{t}_2)$$

Now, eq.(4.1) yields the variable $\bar{\bar{\eta}}$ for the inflationary stage

$$\bar{\bar{\eta}} = \eta_i + \frac{1}{(Q+1)\bar{\bar{A}}_I}\left[(\bar{\bar{t}}_I - \bar{\bar{t}}_i)^{Q+1} - (\bar{\bar{t}}_I - \bar{\bar{t}})^{Q+1}\right] \qquad \bar{\bar{t}}_i < \bar{\bar{t}} < \bar{t}_1 \tag{4.2}$$

For the radiation dominated stage, the corresponding conformal time variable $\bar{\eta}$ takes the form

$$\bar{\eta} = \eta_1 + \frac{1}{(1-R)\bar{A}_{II}}\left[(\bar{t} - \bar{t}_{II})^{1-R} - (\bar{t}_1 - \bar{t}_{II})^{1-R}\right] \qquad \bar{t}_1 < \bar{t} < \bar{t}_2 \tag{4.3}$$

Finally, the variable η in the matter dominated stage is:

$$\eta \;=\; \eta_2 \;+\; \frac{1}{(1-M)A_{III}}\left[t^{1-M} - \bar{t_2}^{\,1-M}\right] \qquad\qquad \bar{t_2} < t < t_0 \tag{4.4}$$

By inverting the above relations eqs.(4.2-4.4) we obtain in the inflationary stage:

$$\bar{\bar{t}} \;=\; \bar{\bar{t}}_I - \left[(\bar{\bar{t}}_I - \bar{\bar{t}}_i)^{Q+1} \;-\; (\bar{\bar{\eta}} - \eta_i)(Q+1)\bar{\bar{A}}_I\right]^{\frac{1}{Q+1}}$$

in the radiation stage

$$\bar{t} \;=\; \bar{t_{II}} \;+\; \left[(\bar{t_1} - \bar{t_{II}})^{1-R} \;+\; (\bar{\eta} - \eta_1)(1-R)\bar{A_{II}}\right]^{\frac{1}{1-R}}$$

and in the matter dominated stage

$$t \;=\; \left[\bar{t_2}^{\,1-M} \;+\; (\eta - \eta_2)(1-M)A_{III}\right]^{\frac{1}{1-M}}$$

With suitable rearrangements of constants, we substitute the above expressions on scale factor eqs.(3.12) and we have the following minimal model written in conformal time:

$$\begin{aligned} \bar{\bar{a}}_I(\bar{\bar{\eta}}) &= \alpha_I(\eta_I - \bar{\bar{\eta}})^{-q} \\ \bar{a_{II}}(\bar{\eta}) &= \alpha_{II}(\eta_{II} + \bar{\eta})^{r} \\ a_{III}(\eta) &= \alpha_{III}(\eta_{III} + \eta)^{m} \end{aligned} \tag{4.5}$$

where the exponents of temporal dependences q, r, m have the following expressions:

$$q = \frac{Q}{Q+1} \quad , \quad r = \frac{R}{1-R} \quad , \quad m = \frac{M}{1-M} \tag{4.6}$$

and the parameters α_j, η_j $(j = I, II, III)$ are:

$$\begin{aligned} \alpha_I &= \left(\bar{\bar{A}}_I(Q+1)^{-Q}\right)^{\frac{1}{Q+1}} \quad , \quad \eta_I = \eta_i + \frac{(\bar{\bar{t}}_I - \bar{\bar{t}}_i)^{Q+1}}{(Q+1)\bar{\bar{A}}_I} \\ \alpha_{II} &= \left(\bar{A_{II}}(1-R)^{R}\right)^{\frac{1}{1-R}} \quad , \quad \eta_{II} = \frac{(\bar{t_1} - \bar{t_{II}})^{1-R}}{(1-R)\bar{A_{II}}} \;-\; \eta_1 \\ \alpha_{III} &= \left(A_{III}(1-M)^{M}\right)^{\frac{1}{1-M}} \quad , \quad \eta_{III} = \frac{(\bar{t_2})^{1-M}}{(1-M)A_{III}} \;-\; \eta_2 \end{aligned} \tag{4.7}$$

Expressions (4.7) are the most general relations among parameters of a scale factor description in conformal time and the corresponding one in cosmic time. No assumptions and no use of continuity conditions have been yet made. When the scale factor in cosmic time description satisfies continuity conditions, their properties eqs.(3.13) allow us to write the relations for α_j, η_j as function of a more reduced number of parameters and transition times $\bar{t_1}$ and $\bar{t_2}$:

$$\alpha_I = \left(\bar{A_{II}}\left(\frac{Q}{R}\right)^Q (Q+1)^{-Q} \left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{R+Q}\right)^{\frac{1}{Q+1}} \tag{4.8}$$

$$\eta_I = \eta_i + \frac{\left(\bar{t_1}(1+\frac{Q}{R}) - (\frac{Q}{R} - \frac{Q}{M})\bar{t_2} - \bar{\bar{t}}_i\right)^{Q+1}}{(Q+1)\bar{A_{II}}(\frac{Q}{R})^Q\left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{R+Q}}$$

$$\alpha_{II} = \left(\bar{A_{II}}(1-R)^R\right)^{\frac{1}{1-R}} \quad , \quad \alpha_{III} = \left(\bar{A_{II}}(1-M)^M\left(\frac{R}{M}\right)^R \ \bar{t_2}^{R-M}\right)^{\frac{1}{1-M}}$$

$$\eta_{II} = \frac{\left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{1-R}}{(1-R)\bar{A_{II}}} - \eta_1 \quad , \quad \eta_{III} = \frac{\bar{t_2}^{1-R}}{(1-M)\bar{A_{II}}(\frac{R}{M})^R} - \eta_2$$

meanwhile the conformal time variables (4.2-4.4) take the next expressions:

$$\bar{\bar{\eta}} = \eta_i + \left[\frac{\left(\bar{t_1}(1+\frac{Q}{R}) - \bar{t_2}(\frac{Q}{R} - \frac{Q}{M}) - \bar{\bar{t}}_i\right)^{Q+1}}{(Q+1)\left(\frac{Q}{R}\right)^Q\bar{A_{II}}\left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{R+Q}} + \right. \tag{4.9}$$

$$\left. - \frac{\left(\bar{t_1}(1+\frac{Q}{R}) - \bar{t_2}(\frac{Q}{R} - \frac{Q}{M}) - \bar{\bar{t}}\right)^{Q+1}}{(Q+1)\left(\frac{Q}{R}\right)^Q\bar{A_{II}}\left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{R+Q}}\right] \qquad \bar{\bar{t}}_i < \bar{\bar{t}} < \bar{t_1}$$

$$\bar{\eta} = \eta_1 + \frac{\left[\left(\bar{t} - (1-\frac{R}{M})\bar{t_2}\right)^{1-R} - \left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{1-R}\right]}{(1-R)\bar{A_{II}}} \qquad \bar{t_1} < \bar{t} < \bar{t_2}$$

$$\eta = \eta_2 + \frac{\left[t^{1-M} - \bar{t_2}^{1-M}\right]}{(1-M)\bar{A_{II}}(\frac{R}{M})^R \ (\bar{t_2})^{R-M}} \qquad \bar{t_2} < t < t_0$$

At this point, we have not considered whatever or not the conformal time variable itself is continuous. But as consequence of the continuity of the scale factor written in cosmic time-type variables along $\bar{t_1}$ and $\bar{t_2}$, we have yet:

$$\left.\frac{d\bar{\bar{\eta}}}{d\bar{\bar{t}}}\right|_{\bar{t_1}^-} = \left.\frac{d\bar{\eta}}{d\bar{t}}\right|_{\bar{t_1}^+} = \frac{1}{\bar{A_{II}}\left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^R} \tag{4.10}$$

$$\left.\frac{d\bar{\eta}}{d\bar{t}}\right|_{\bar{t_2}^-} = \left.\frac{d\eta}{dt}\right|_{\bar{t_2}^+} = \frac{\left(\frac{R}{M}\bar{t_2}\right)^{-R}}{\bar{A_{II}}}$$

4.2 The continuous η case

Now, we can impose conditions relating the three conformal time variables eqs.(4.9) since we have the freedom in the parameters defined as η_i, η_1 and η_2.

We will constrain the conformal time to be continuous along the transitions at $\bar{t_1}$ and $\bar{t_2}$, where the scale factor written in cosmic time-type variables have fully satisfactories continuity conditions. In this way, we will have the conformal time univocally defined at these transitions.

$$\begin{aligned} \eta_1 = \bar{\eta}(\bar{t_1}) \quad &\equiv \quad \bar{\bar{\eta}}(\bar{t_1}) \\ \eta_2 = \eta(\bar{t_2}) \quad &\equiv \quad \bar{\eta}(\bar{t_2}) \end{aligned} \tag{4.11}$$

These conditions and eqs.(4.2-4.4) give two relations linking the parameters η_i and the two transitions η_1 and η_2:

$$\eta_2 = \eta_1 \; + \; \frac{1}{(1-R)\bar{A_{II}}} \left[(\bar{t_2} - \bar{t_{II}})^{1-R} - (\bar{t_1} - \bar{t_{II}})^{1-R} \right] \tag{4.12}$$

$$\eta_1 = \eta_i \; + \; \frac{1}{(Q+1)\bar{\bar{A}}_I} \left[(\bar{\bar{t}}_I - \bar{\bar{t}}_i)^{Q+1} - (\bar{\bar{t}}_I - \bar{t_1})^{Q+1} \right] \tag{4.13}$$

Making use of the continuity properties of scale factor in cosmic time description (2.2) we can write eq.(4.13) as:

$$\eta_1 = \eta_i \; + \; \frac{\frac{Q}{R}(\bar{t_1} - \bar{t_{II}})^{1-R}}{(Q+1)\bar{A_{II}}} \left[\left(1 + \frac{\bar{t_1} - \bar{\bar{t}}_i}{\frac{Q}{R}(\bar{t_1} - \bar{t}_{II})} \right)^{Q+1} \; - \; 1 \right] \tag{4.14}$$

and by using the matching eqs.(3.15) and relations (3.14) we can express them as functions of the observational transition times t_r and t_m:

$$\begin{aligned} \eta_2 &= \eta_1 \; + \; \frac{\left(\frac{R}{M}\right)^{1-R}}{(1-R)\bar{A_{II}}} ({t_m}^{1-R} - {t_r}^{1-R}) \\ \eta_1 &= \eta_i \; + \; \frac{\frac{Q}{R}\left(\frac{R}{M} t_r\right)^{1-R}}{(Q+1)\bar{A_{II}}} \left[\left(\frac{t_i - t_I}{t_r - t_I} \right)^{Q+1} \; - \; 1 \right] \end{aligned} \tag{4.15}$$

The remaining freedom is used by convenience to eliminate the parameter η_{II} in order to simplify the expression of the radiation dominated stage scale factor: $\eta_{II} = 0$. From (4.7) is immediately obtained:

$$\eta_1 = \frac{(\bar{t_1} - \bar{t_{II}})^{1-R}}{(1-R)\bar{A_{II}}} \tag{4.16}$$

thus, we have η_1 as a parameter univocally fixed in terms of parameters of the minimal model scale factor. With eq.(4.12) and (4.16) we obtain the fixed expression for η_2:

$$\eta_2 = \frac{(\bar{t_2} - \bar{t_{II}})^{1-R}}{(1-R)\bar{A_{II}}} \tag{4.17}$$

and the parameter η_i is equally fixed in terms of the cosmic time-type variables from eqs.(4.14) and (4.16).

$$\eta_i = \frac{(\bar{t_1} - \bar{t_{II}})^{1-R}}{(Q+1)\bar{A_{II}}} \left[\frac{R+Q}{R(1-R)} - \frac{Q}{R}\left(1 + \frac{\bar{t_1} - \bar{\bar{t_i}}}{\frac{Q}{R}(\bar{t_1} - \bar{t_{II}})}\right)^{Q+1} \right] \tag{4.18}$$

Using again the matching expressions eqs.(3.15) and the relations (3.14) on eqs.(4.16-4.18) we can finally write the expressions relating the conformal time transitions η_1, η_2 and η_i with the observational standard values for the transitions at the beginning of radiation dominated stage t_r, beginning of matter dominated stage t_m and beginning of inflationary stage t_i:

$$\begin{aligned} \eta_1 &= \frac{\left(\frac{R}{M}\, t_r\right)^{1-R}}{(1-R)\bar{A_{II}}} \quad , \quad \eta_2 = \frac{\left(\frac{R}{M}\, t_m\right)^{1-R}}{(1-R)\bar{A_{II}}} \\ \eta_i &= \frac{\left(\frac{R}{M}\, t_r\right)^{1-R}}{(Q+1)\bar{A_{II}}} \left[\frac{R+Q}{R(1-R)} - \frac{Q}{R}\left(\frac{t_i - t_I}{t_r - t_I}\right)^{Q+1} \right] \end{aligned} \tag{4.19}$$

Eqs.(4.16-4.18) or the corresponding ones in terms of observational transition times eqs.(4.19) constitute the continuity and simplicity conditions for the conformal time variable chosen. From eqs.(4.9) with help of eqs.(3.14),(3.15) and (4.19), the continuous conformal time as function of transition times t_r and t_m and proper cosmic time t in each stage is:

$$\begin{aligned} \bar{\bar{\eta}} &= \frac{(\frac{R}{M} t_r)^{1-R}}{\bar{A_{II}}(Q+1)} \left[\frac{R+Q}{R(1-R)} - \frac{Q}{R}\left(\frac{t_I - t}{t_I - t_r}\right)^{Q+1} \right] & t_i < t < t_r \\ \bar{\eta} &= \frac{(\frac{R}{M} t)^{1-R}}{\bar{A_{II}}(1-R)} & t_r < t < t_m \\ \eta &= \frac{(\frac{R}{M} t_m)^{1-R}}{\bar{A_{II}}(1-M)} \left[\frac{R-M}{R(1-R)} + \frac{M}{R}\left(\frac{t}{t_m}\right)^{1-M} \right] & t_m < t < t_0 \end{aligned} \tag{4.20}$$

With this continuous conformal time (4.20), the scale factor takes the form:

$$\begin{aligned} \bar{\bar{a}}_I(\bar{\bar{\eta}}) &= \alpha_I(\eta_I - \bar{\bar{\eta}})^{-q} & \eta_i < \bar{\bar{\eta}} < \eta_1 \\ \bar{a_{II}}(\bar{\eta}) &= \alpha_{II}(\bar{\eta})^r & \eta_1 < \bar{\eta} < \eta_2 \\ a_{III}(\eta) &= \alpha_{III}(\eta_{III} + \eta)^m & \eta_2 < \eta \end{aligned} \tag{4.21}$$

where the parameters q, r and m obey again relations (4.6) and the coefficients α_j and η_j ($j = I, II, III$) have these expressions since the continuity conditions (4.19) hold:

$$\alpha_I = \left(\bar{A_{II}}\left(\frac{R}{M}\right)^R\left(\frac{Q}{M}\right)^Q\frac{1}{(Q+1)^Q}\ t_r{}^{R+Q}\right)^{\frac{1}{Q+1}} \tag{4.22}$$

$$\eta_I = \frac{R+Q}{R(Q+1)(1-R)}\frac{(\frac{R}{M}t_r)^{1-R}}{\bar{A_{II}}}$$

$$\alpha_{II} = \bar{A_{II}}^{\frac{1}{1-R}}(1-R)^{\frac{R}{1-R}} \quad , \quad \alpha_{III} = \left(\bar{A_{II}}(1-M)^M\left(\frac{R}{M}\right)^R t_m{}^{R-M}\right)^{\frac{1}{1-M}}$$

$$\eta_{II} = 0 \qquad , \qquad \eta_{III} = \frac{M-R}{R(1-M)(1-R)}\frac{(\frac{R}{M}t_m)^{1-R}}{\bar{A_{II}}}$$

4.3 Continuity of the scale factor in conformal time

The relations (4.21) and (4.22) have been obtained by imposing to the conformal time variable itself some conditions of simplicity and continuity at transtitions η_1 and η_2. The same results could be reached by imposing equivalent relations to the scale factor written in conformal time.

We can take the more general case for the scale factor eqs.(4.5) and impose it to have transitions at generic parameters η_1 and η_2 with continuity for the scale factor and its first derivative with respect to conformal time. This matching led to the next four relations between the six parameters of the scale factor α_j, η_j ($j = I, II, III$):

$$\alpha_I = \alpha_{II}\left(\frac{q}{r}\right)^q(\eta_1 + \eta_{II})^{r+q} \tag{4.23}$$

$$\eta_I = \eta_1\left(1 + \frac{q}{r}\right) + \frac{q}{r}\ \eta_{II} \tag{4.24}$$

$$\alpha_{III} = \alpha_{II}\left(\frac{m}{r}\right)^{-m}(\eta_{II} + \eta_2)^{r-m} \tag{4.25}$$

$$\eta_{III} = \eta_2\left(\frac{m}{r} - 1\right) + \frac{m}{r}\ \eta_{II} \tag{4.26}$$

Now, we find the conditions to be satisfied by the parameters η_1 and η_2 in order to satisfy the matching conditions eqs.(4.23-4.26) when the conformal time variable has been obtained from a cosmic time-type description with continuous and smooth transitions. Thus, we have in principle the general expressions (4.8) for α_j, η_j ($j = I, II, III$). It is straightforward to see that eq.(4.23) is trivially satisfied. From eq.(4.24) we obtain:

$$\eta_i = \eta_1 + \frac{\frac{Q}{R}(\bar{t_1} - (1-\frac{R}{M})\bar{t_2})^{1-R}}{(Q+1)\bar{A_{II}}}\left[1 - \left(1 + \frac{\bar{t_1} - \bar{\bar{t_i}}}{\frac{Q}{R}(\bar{t_1} - (1-\frac{R}{M})\bar{t_2})}\right)^{Q+1}\right] \tag{4.27}$$

Meanwhile, both eqs.(4.25) and (4.26) give the same condition:

$$\eta_2 = \eta_1 \; + \; \frac{1}{(1-R)\bar{A_{II}}}\left[\left(\frac{R}{M}\bar{t_2}\right)^{1-R} \; - \; \left(\bar{t_1} - (1-\frac{R}{M})\bar{t_2}\right)^{1-R}\right] \tag{4.28}$$

It is easy to see that eqs.(4.27) and (4.28) are equivalent to expressions (4.14) and (4.12). By making use of expressions (3.15) and (3.14) we can write them as function of observational values:

$$\eta_2 \; - \; \eta_1 = \frac{(\frac{R}{M})^{1-R}}{(1-R)\bar{A_{II}}}({t_m}^{1-R} - {t_r}^{1-R})$$

$$\eta_i \; - \; \eta_1 = \frac{\frac{Q}{R}\left(\frac{R}{M}t_r\right)^{1-R}}{(Q+1)\bar{A_{II}}}\left[1 \; - \; \left(\frac{t_i - t_I}{t_r - t_I}\right)^{Q+1}\right]$$

That means, we have recovered the conditions (4.15) obtained from imposing continuity to conformal time variable itself before simplifying $\eta_{II} = 0$.

In fact, the four conditions of continuity of the scale factor written in conformal time (4.23) are not independent equations when the conformal time variable is constructed over a continuous description in cosmic time. Continuity in the first derivative of the conformal time scale factor is warranted from continuity in the first derivative of the cosmic time description because equality of the relative derivatives eqs.(4.10). Thus, the only two independent conditions are the equality in the scale factor at the transitions η_1 and η_2. Satisfy them impose not only the continuity in the scale factor written in cosmic time at $\bar{t_1}$ and $\bar{t_2}$ but also the continuity in the conformal time variable itself along the transitions, exactly as made in eqs.(4.11) and remaining always one freedom that we fix by imposing the simplifying condition $\eta_{II} = 0$.

4.4 Summary of the scale factor in conformal time

We have used the scale factor description written in cosmic time-type variables (See eqs.(3.12) and related ones) in order to define a continuous conformal time variable and also the corresponding scale factor with suitable continuity and simplicity properties:

$$\begin{aligned} \bar{\bar{a}}_I(\bar{\bar{\eta}}) &= \alpha_I(\eta_I - \bar{\bar{\eta}})^{-q} & \eta_i < \bar{\bar{\eta}} < \eta_1 \\ \bar{a_{II}}(\bar{\eta}) &= \alpha_{II}(\bar{\eta})^r & \eta_1 < \bar{\eta} < \eta_2 \\ a_{III}(\eta) &= \alpha_{III}(\eta_{III} + \eta)^m & \eta_2 < \eta \end{aligned} \tag{4.29}$$

with continuous, sudden and smooth transitions (continuity of scale factor and first derivative with respect to conformal time) at η_1 and η_2. The parameters α_j, η_j $(j = I, II, III)$ satisfy these matching relations:

$$\begin{aligned} \alpha_I &= \alpha_{II}\left(\frac{q}{r}\right)^q (\eta_1)^{r+q} \quad , \quad \eta_I = \eta_1\left(1+\frac{q}{r}\right) \\ \alpha_{III} &= \alpha_{II}\left(\frac{m}{r}\right)^{-m} (\eta_2)^{r-m} \quad , \quad \eta_{III} = \eta_2\left(\frac{m}{r}-1\right) \end{aligned} \tag{4.30}$$

It must be remembered that all the parameters of this scale factor in conformal time have complete and univocally translation as functions of the observational transition times t_r, t_m, the cosmic time $t = c\mathcal{T}$ and the global scale factor constant $\bar{A_{II}}$. This dictionary is constituted by the expressions given for the exponents q, r and m in eqs.(4.6) and the corresponding ones for α_j, η_j $(j = I, II, III)$ by eqs.(4.22). Remember also the expressions for conformal time variables eqs.(4.20) and the expressions for the transition parameters η_1, η_2 and η_i eqs.(4.19).

In this way, we have obtained a minimal model with continuous, sudden and smooth transitions in cosmic time description (3.12) and also in conformal time variables (4.29).

5 Another Conformal Time Construction

We have used above the descriptive cosmic time variables in order to linking the conformal time variable with the observational information. Another alternative arises: it is possible to link the conformal time variable *directly* with the current cosmic time variable (t) without passing the descriptive variables.

This possibility will give us a conformal time description *closer* to cosmic time and observational information. So close that it will hereditate the lacking of full satisfactory continuity conditions, as we will see. Differently from the former one, this conformal time variable satisfies continuity itself in the proper time derivatives. Transitions are studied directly on the observational values t_r and t_m instead of the descriptive ones $\bar{t_1}$, $\bar{t_2}$ where fully continuity could be introduced. Scale factor in this conformal time variable is not as continuous as last case. Their transitions can be made sudden and continuous, but never smooth. Matching relations for this scale factor are differents from the above given. For studies where full continuity of scale factor can be overcome or closeness to observational information is prioritary, this approach could result useful.

Again, we will have a particular conformal time variable for each stage, oughted to the different shape of the scale factor . We define:

$$
\begin{aligned}
d\bar{\bar{\eta}} &= \frac{dt'}{\bar{\bar{a}}_I(t')} \qquad t_i < t' < t_r \\
d\bar{\eta} &= \frac{dt'}{\bar{a}_{II}(t')} \qquad t_r < t' < t_m \\
d\eta &= \frac{dt'}{a_{III}(t')} \qquad t_m < t' < t_0
\end{aligned}
\tag{5.1}
$$

where $a_I(t)$, $a_{II}(t)$ and $a_{III}(t)$ are given by eqs.(3.16). Working as made in the general conformal time case, we integrate the equations above and obtain expressions for the conformal time variable in each stage. Thus, we define the variable $\bar{\bar{\eta}}$ for the inflationary stage, $\bar{\eta}$ for radiation dominated stage and η for matter dominated stage:

$$
\begin{aligned}
\bar{\bar{\eta}} &= \eta_i + \frac{\left(\frac{t_I}{t_r} - 1\right)^{-Q}}{(Q+1)\bar{A}_{II} {t_r}^{R+Q} (\frac{R}{M})^R} \left[(t_I - t_i)^{Q+1} - (t_I - t)^{Q+1} \right] \qquad t_i < t < t_r \\
\bar{\eta} &= \eta_1 + \frac{1}{(1-R)\bar{A}_{II}(\frac{R}{M})^R} (t^{1-R} - {t_r}^{1-R}) \qquad t_r < t < t_m \\
\eta &= \eta_2 + \frac{1}{(1-M)\bar{A}_{II} {t_m}^{R-M} (\frac{R}{M})^R} (t^{1-M} - {t_m}^{1-M}) \qquad t_m < t < t_0
\end{aligned}
$$

where we have defined this time the parameters $\eta_i = \bar{\bar{\eta}}(t_i)$, $\eta_1 = \bar{\eta}(t_r)$ and $\eta_2 = \eta(t_m)$. By inverting the above relations for η, $\bar{\eta}$ and $\bar{\bar{\eta}}$ we obtain expressions for t in each one of the stages:

$$
\begin{aligned}
t_I - t &= \left[(t_I - t_i)^{Q+1} - (\bar{\bar{\eta}} - \eta_i)(Q+1)\bar{A}_{II} \left(\frac{R}{M}\right)^R {t_r}^{R+Q} \left(\frac{t_I}{t_r} - 1\right)^Q \right]^{\frac{1}{Q+1}} \quad t_i < t < t_r \\
t &= \left[{t_r}^{1-R} + (\bar{\eta} - \eta_1)(1-R)\bar{A}_{II} \left(\frac{R}{M}\right)^R \right]^{\frac{1}{1-R}} \qquad t_r < t < t_m \\
t &= \left[{t_m}^{1-M} + (\eta - \eta_2)(1-M)\bar{A}_{II} \left(\frac{R}{M}\right)^R {t_m}^{R-M} \right]^{\frac{1}{1-M}} \quad t_m < t < t_0
\end{aligned}
$$

With some straightforward computations and suitable rearrangements of constants, we obtain expressions for the scale factor as seen in eqs.(4.5) whose temporal dependence exponentes are given again by eqs.(4.6) and the parameters α_j, η_j ($j = I, II, III$) are:

$$\alpha_I = \left(\bar{A_{II}}\left(\frac{R}{M}\right)^R \left(\frac{t_I}{t_r} - 1\right)^Q (Q+1)^{-Q} \;\; {t_r}^{R+Q}\right)^{\frac{1}{Q+1}} \tag{5.2}$$

$$\eta_I = \eta_i + \frac{1}{(Q+1)\bar{A_{II}}{t_r}^{R+Q}(\frac{R}{M})^R}\left(\frac{t_I}{t_r} - 1\right)^{-Q}(t_I - t_i)^{Q+1}$$

$$\alpha_{II} = \left(\bar{A_{II}}(1-R)^R\left(\frac{R}{M}\right)^R\right)^{\frac{1}{1-R}} \quad , \quad \alpha_{III} = \left(\bar{A_{II}}(1-M)^M\left(\frac{R}{M}\right)^R \;\; {t_m}^{R-M}\right)^{\frac{1}{1-M}}$$

$$\eta_{II} = \frac{{t_r}^{1-R}}{(1-R)\bar{A_{II}}(\frac{R}{M})^R} - \eta_1 \quad , \quad \eta_{III} = \frac{{t_m}^{1-R}}{(1-M)\bar{A_{II}}(\frac{R}{M})^R} - \eta_2$$

Now, we can impose conditions relating the three conformal time variables in order to eliminate the freedom in the parameters defined as η_i, η_1 and η_2. We use it for constraining the conformal time to be continuous along the transitions at t_m and t_r (and, as consequence, univocally defined at these transitions).

$$\eta_1 = \bar{\eta}(t_r) \;\; \equiv \;\; \bar{\bar{\eta}}(t_r)$$
$$\eta_2 = \eta(t_m) \;\; \equiv \;\; \bar{\eta}(t_m)$$

and the remaining freedom is used again in eliminate the parameter η_{II}, simplifying thus the expression of the radiation dominated stage scale factor $\eta_{II} = 0$. Whit these three conditions, we can fix the parameters η_i, η_1 and η_2 as functions of parameters of the minimal model only. The parameter η_i is:

$$\eta_i = \frac{1}{{t_r}^R\bar{A_{II}}(\frac{R}{M})^R}\left[\frac{t_r}{(1-R)} - \frac{(t_I - t_r)}{(Q+1)}\left[\left(\frac{t_I - t_i}{t_I - t_r}\right)^{Q+1} - 1\right]\right] \tag{5.3}$$

and the parameters related to the transitions are:

$$\eta_1 = \frac{{t_r}^{1-R}}{\bar{A_{II}}(1-R)(\frac{R}{M})^R} \quad , \quad \eta_2 = \frac{{t_m}^{1-R}}{\bar{A_{II}}(1-R)(\frac{R}{M})^R} \tag{5.4}$$

In this way, the conformal time variables are continuous and univocally defined at transitions η_1 and η_2. In terms of minimal model parameters, they are:

$$\bar{\bar{\eta}} = \frac{(t_r)^{-R}}{\bar{A_{II}}\left(\frac{R}{M}\right)^R(Q+1)}\left[\frac{R+Q}{(1-R)}\; t_r + t_I - \frac{(t_I - t)^{Q+1}}{(t_I - t_r)^Q}\right] \tag{5.5}$$

$$\bar{\eta} = \frac{(t)^{1-R}}{\bar{A_{II}}(1-R)\left(\frac{R}{M}\right)^R}$$

$$\eta = \frac{(t_m)^{1-R}}{\bar{A_{II}}(1-M)\left(\frac{R}{M}\right)^R}\left[\frac{R-M}{1-R} + \left(\frac{t}{t_m}\right)^{1-M}\right]$$

Notice this conformal time variable enjoys continuity along the transitions and observational times t_r and t_m, not only on the variable itself but also in their derivative with respect to proper cosmic time in each stage:

$$\left.\frac{d\bar{\bar{\eta}}}{dt}\right|_{t_r^-} = \left.\frac{d\bar{\eta}}{dt}\right|_{t_r^+} = \frac{{t_r}^{-R}}{\bar{A_{II}}\left(\frac{R}{M}\right)^R} \tag{5.6}$$
$$\left.\frac{d\bar{\eta}}{dt}\right|_{t_m^-} = \left.\frac{d\eta}{dt}\right|_{t_m^+} = \frac{{t_m}^{-R}}{\bar{A_{II}}\left(\frac{R}{M}\right)^R}$$

With these variables, we obtain the model written in continuous conformal time as can be seen in eqs.(4.21). In this case, its parameters take this form:

$$\alpha_I = \left(\bar{A_{II}}\left(\frac{R}{M}\right)^R\left(\frac{t_I}{t_r}-1\right)^Q(Q+1)^{-Q}\ \ {t_r}^{R+Q}\right)^{\frac{1}{Q+1}} \tag{5.7}$$
$$\eta_I = \frac{{t_r}^{1-R}}{\bar{A_{II}}(Q+1)\left(\frac{R}{M}\right)^R}\left[\frac{t_I}{t_r}+\frac{Q+R}{1-R}\right]$$
$$\alpha_{II} = \left(\bar{A_{II}}(1-R)^R\left(\frac{R}{M}\right)^R\right)^{\frac{1}{1-R}} \quad , \quad \alpha_{III} = \left(\bar{A_{II}}(1-M)^M\left(\frac{R}{M}\right)^R{t_m}^{R-M}\right)^{\frac{1}{1-M}}$$
$$\eta_{II} = 0 \quad , \quad \eta_{III} = \frac{M-R}{(1-M)(1-R)}\ \frac{(t_m)^{1-R}}{\bar{A_{II}}\left(\frac{R}{M}\right)^R}$$

Notice that parameters (5.7) do not satisfy eqs.(4.30). That means, the scale factor (4.29) does not have sudden, continuos and smooth transitions among stages when it runs on this conformal time variables (5.5). Scale factor is continuous, but not thus its first derivative with respect to conformal time. In this case, we have sudden and continuous transitions at η_1 and η_2, but they are not smooth.

In fact, we have continuity in the scale factor $\bar{\bar{a}}_I(\eta_1) = \bar{a_{II}}(\eta_1)$ and $\bar{a_{II}}(\eta_2) = a_{III}(\eta_2)$. For the first derivative, instead of equality among both sides of transition, we have:

$$\left.\frac{d\bar{\bar{a}}_I(\bar{\bar{\eta}})}{d\bar{\bar{\eta}}}\right|_{\eta_1^-} = \frac{R}{Q}\left(\frac{t_I}{t_r}-1\right)\left.\frac{d\bar{a_{II}}(\bar{\eta})}{d\bar{\eta}}\right|_{\eta_1^+}$$
$$\left.\frac{d\bar{a_{II}}(\bar{\eta})}{d\bar{\eta}}\right|_{\eta_2^-} = \frac{M}{R}\left.\frac{da_{III}(\eta)}{d\eta}\right|_{\eta_2^+}$$

Explanation is simple: this conformal time variable maps very closer the cosmic proper times. So closer that equality in its derivatives with respect to proper cosmic times hold along the transitions. But we know there is an inherent discontinuity in the proper cosmic time derivatives of scale factor, as discussed above

(See eqs.(3.16-3.17) and comments below). This discontinuity was absorbed in the descriptive cosmic time variables, as can be seen from their derivatives eqs.(3.18). Lacking of this descriptive variables, this construction of conformal time is not able to absorb such discontinuity at transitions t_r and t_m and the related proportionality factors (3.18) reappear in the conformal time derivatives of scale factor.

By arranging in the right way the equality among the right-hand and left-hand derivatives, the matching equations for the scale factor (4.29) with conformal time (5.5) take the form:

$$\alpha_I = \alpha_{II}(\eta_1)^{r+q}\left(\frac{q}{r}\right)^q\left(\frac{Q}{R}\frac{t_r}{t_I - t_r}\right)^{-q} \quad , \quad \eta_I = \eta_1\left(1 + \frac{q}{r}\left(\frac{Q}{R}\frac{t_r}{t_I - t_r}\right)^{-1}\right) \tag{5.8}$$

$$\alpha_{III} = \alpha_{II}{\eta_2}^{r-m}\left(\frac{m}{r}\frac{R}{M}\right)^{-m} \quad , \quad \eta_{III} = \eta_2\left(\frac{m}{r}\frac{R}{M} - 1\right)$$

6 Conclusions

Our starting point and motivation was to realize that modelized transitions can not reproduce real transitions. We point out that in cosmologies dealing with discountinuous stages describing a step-by-step evolution for the scale factor without underlying satisfactory description of transitions, is not enough to match the scale factor. At priori, the spacetime metric can be discontinuous, leading to discontinuities in the temporal variables itself.

As the description of different stages come from different coordinate systems, there exists the possibility of transformations relating them. We have used this freedom choosing coordinates with satisfactory continuous matching along transitions and containing the minimal information about the observational Universe.

The minimal observational information are the transition time scales. It gives us scale factor ratii reached in each evolution stage as a powerful tool. All our constructions are constrained to reach the same expansion ratii in each one of the stages eqs.(3.2-3.4).

The set of parameters relating descriptive scale factor and observational Universe information is given. This scale factor (3.12) runs on transformed variables ((3.9),(3.11)) and has modified durations on inflationary and radiation dominated stages. The matter dominated stage is considered untransformed in the description, since observations must be made from this current stage. Modelized transitions happen at transformed temporal coordinates, related with the observational values (3.14).

Also, a set of parameters giving satisfactory continuous and predictive conformal time minimal model is computed. The conformal time variable (4.20) is constructed on the descriptive model and constrained to be continuous along the transformed transitions (4.19). A remaining freedom is invested in simplicity for the corresponding scale factor expressions (4.29).

The minimal model thus obtained have continuous, sudden and smooth transitions both in cosmic time and in conformal time descriptions. Minimal information of observational Universe have been introduced in both cases. Parameters and variables are linked with observational standard values and current proper cosmic time in a totally determined way ((3.15),(4.6),(4.30) and related expressions). It preserves the physical meaning and predictability of computations made with these constructions and allows them to be confronted with measurementes and observations. For instance, gravitational wave generation and related type computations can benefit of this kind of treatment.

We have included also another conformal time variable with different continuity and predictive characteristics. It is constructed on the proper cosmic time directly and maps it closer than the former one (5.5). This variable is smoothly continuous along the transitions at observational times. In opposite way, the scale factor running on it presents sudden and continuous transitions, but never smooth. It is oughted to the lack of descriptive intermediate variables absorbing the inherent discontinuity in the proper time derivatives of scale factor at real transitions.

Of course, we do not consider this treatment solves the problem of transitions among the stages for which a full dynamical description is required. We recall the problem of continuity of temporal coordinate variable in step-by-step cosmologies and we suggest some ways in order to overcome it, at least useful in extraction of some observational consequences. It is clearly useful in metric perturbations type computations, and it can be applied in whatever subject involving different cosmological evolution stges. It opens the way to new and more exact treatments.

Appendixes

A Scale Factor Description for Power Law Inflationary Stages

In the last sections we have treated a minimal model including an inverse power inflationary stage. We recall that this process is easily genralized to other type of inflationary stages. In sake of completeness, we report here the results and link with observational information for a model including an arbitrary power law inflationary stage. Notice some differences in the inflationary scale factor would modify expressions in the linking process.

The minimal model to be linked with observational information would be in this case:

$$\begin{aligned} \bar{\bar{a}}_I(\bar{\bar{t}}) &= \bar{\bar{A}}_I(\bar{\bar{t}} - \bar{\bar{t}}_I)^{-Q} && \bar{\bar{t}} < \bar{t}_1 \\ \bar{a}_{II}(\bar{t}) &= \bar{A}_{II}(\bar{t} - \bar{t}_{II})^{R} && \bar{t}_1 < \bar{t} < \bar{t}_2 \\ a_{III}(t) &= A_{III}(t - t_{III})^{M} && \bar{t}_2 < t \end{aligned} \tag{A.1}$$

The inflationary stage (I) is an accelerated expansion provided $Q > 1$ and $t_I < t_i < t_r$. The usual power law is found for $t_I = 0$. The same behaviours before considered for radiation and matter dominated stages hold. Again, we impose this model to have continuous transitions at $\bar{t}_1$ and $\bar{t}_2$ both for scale factor and first derivative. The matching relations thus obtained are:

$$\begin{aligned} \bar{\bar{t}}_I &= \bar{t}_1\left(1 - \frac{Q}{R}\right) + \bar{t}_{II}\frac{Q}{R} \quad , \quad t_{III} = \left(1 - \frac{M}{R}\right)\bar{t}_2 + \frac{M}{R}\bar{t}_{II} \\ \bar{\bar{A}}_I &= \bar{A}_{II}\left(\frac{R}{Q}\right)^{Q}(\bar{t}_1 - \bar{t}_{II})^{R-Q} \quad , \quad A_{III} = \bar{A}_{II}\left(\frac{R}{M}\right)^{M}(\bar{t}_2 - \bar{t}_{II})^{R-M} \end{aligned} \tag{A.2}$$

In the linking with observational Universe information, we apply again the relations (3.2-3.4) and fix also temporal coordinate in the third stage with the corresponding proper cosmic time as made in (3.5). With this, we obtain the expressions for the transitions:

$$\begin{aligned} \bar{t}_1 &= \frac{R}{M}t_r + \left(1 - \frac{R}{M}\right)t_m \quad , \quad \bar{t}_2 = t_m \\ \bar{\bar{t}}_i &= \left(\frac{R}{M} - \frac{Q}{M}\frac{t_r - t_i}{t_r - t_I}\right)t_r + \left(1 - \frac{R}{M}\right)t_m \end{aligned} \tag{A.3}$$

and the descriptive variables are:

$$\begin{aligned} \bar{\bar{t}} &= \left(\frac{R}{M} - \frac{Q}{M}\frac{t_r - t}{t_r - t_I}\right)t_r + \left(1 - \frac{R}{M}\right)t_m \\ \bar{t} &= \frac{R}{M}t + \left(1 - \frac{R}{M}\right)t_m \end{aligned} \tag{A.4}$$

In terms of observational transition values t_r and t_m, the parameters of the scale factor description take the form:

$$\begin{aligned} \bar{\bar{t}}_I &= t_r\left(\frac{R}{M} - \frac{Q}{M}\right) + t_m\left(1 - \frac{R}{M}\right) \quad , \quad \bar{t}_{II} = \left(1 - \frac{R}{M}\right)t_m \\ \bar{\bar{A}}_I &= \bar{A}_{II}\left(\frac{M}{Q}\right)^{Q}\left(\frac{R}{M}\right)^{R}{t_r}^{R-Q} \quad , \quad A_{III} = \bar{A}_{II}\left(\frac{R}{M}\right)^{R}{t_m}^{R-M} \end{aligned} \tag{A.5}$$

In this case, the duration of stages is again modified in the descriptive variables. We have matter dominated hold the same duration. For radiation dominated and inflationary stages we have:

$$\Delta\bar{t}|_{RAD} = \frac{R}{M}\,\Delta t|_{RAD}$$
$$\Delta\bar{\bar{t}}\Big|_{INF} = \frac{Q}{M}\left(1+\frac{t_I}{t_r-t_I}\right)\Delta t|_{INF}$$

In this case, the radiation dominated stage is again contracted in the descriptive variables. For inflationary stage, a dilatation takes place in whatever case. For pure power law, we have $\Delta\bar{\bar{t}} = \frac{Q}{M}\Delta t$ where $\frac{Q}{M} > 1$ for inflationary behaviours.

B An Example of Intermediate Transition Stage

As an example, we present a more realistic alternative to the transitions here supposed sudden. We attempt to elaborate a continuous and smooth evolution of scale factor running on a proper cosmic time coordinate fixed before, along and after the transition.

Let us consider an intermediate stage between the radiation dominated and matter dominated ones. As above said, it is not possible a sudden and smoothly continuous transition there. We relaxe these assumptions by considering the intermediate brief stage $a_{tr}(t)$ with variable evolution law $N(t)$ matching continuous and smoothly with the radiation stage at t_m and with the matter stage at $t_m+\Delta$:

$$a_{II}(t) = A_{II}t^R \qquad a_{tr}(t) = A_{II}t^{N(t)} \qquad a_{III}(t) = t^M$$

In the exit of radiation dominated stage t_m, the conditions to be satisfied are:

$$a_{II}(t_m) = a_{tr}(t_m) \quad , \quad \dot{a_{II}}(t)|_{t_m} = \dot{a_{tr}}(t)|_{t_m} \tag{B.1}$$

where dot means derivative with respecto to the fixed proper cosmic time coordinate t. In the beginning of matter dominated stage at $t_m+\Delta$ we have:

$$a_{tr}(t_m+\Delta) = a_{III}(t_m+\Delta) \quad , \quad \dot{a_{tr}}(t)|_{t_m+\Delta} = \dot{a_{III}}(t)|_{t_m+\Delta} \tag{B.2}$$

From eqs.(B.1), we obtain two boundary conditions for the function $N(t)$ in the intermediate stage: $N(t_m) = R$ and $\dot{N(t)}\Big|_{t_m} = 0$. From eqs.(B.2), we extract a relation among the constants of global scale factors:

$$A_{III} = A_{II}(t_m+\Delta)^{N(t_m+\Delta)-M}$$

Notice this is a different relation from obtained in the treatment with descriptive variables eq.(3.15). Finally, the remaining information is:

$$\dot{N(t)}\Big|_{t_m+\Delta} \ln\left(t_m+\Delta\right) = \left(M - N\left(t_m+\Delta\right)\right)\left(t_m+\Delta\right)^{-1} \tag{B.3}$$

Without further information about Δ, this equation gives us as possible solution: $N\left(t_m+\Delta\right) = M,\ \dot{N(t)}\Big|_{t_m+\Delta} = 0$ and $A_{III} = A_{II}$. Thus, $N(t)$ would be a function interpolating among values R and M in a brief interval Δ with vanishing derivative in both extremes. This behaviour would mean a not negligible dynamics of scale factor during the transition, with a changing evolution in a short interval.

If we consider an arbitrary function $N(t)$ without vanishing derivatives at the extremes, the eq.(B.3) can be approximated in the case of smooth interpolation leading to the next relation:

$$N\left(t_m+\Delta\right) = R\frac{A}{A+1} + M\frac{1}{1+A} \tag{B.4}$$

where $A = \left(1+\frac{t_m}{\Delta}\right)\ln(t_m+\Delta)$. In the approximation of transition almost sudden, we can take $\Delta \to \delta$ neglecting quadratic terms. In this case, we obtain $N(t_m+\delta) \sim R+(M-R)\frac{\delta}{t_m}\ln(t_m)$ and as consequence $A_{III} \sim A_{II}(t_m+\delta)^{(R-M)(1-\frac{\delta}{t_m}\ln t_m)}$. We recover the expressions like (3.13). Notice in anycase that more complex dynamics are in principle possible and without better comprehension of phenomena driving the transitions, it is not possible to give an unique and satisfactory description of them.

C Scale Factor Description Based on Radiation Dominated Stage

Until now, we have worked with minimal models for scale factor evolution including the current matter dominated stage. Link with observations made from this last stage have advised us to privilege it in conserving its proper cosmic time coordinate in our description. Again, alternatives arise. It is possible privilege the proper cosmic time of radiation dominated stage, if one wants consider only the first stages in minimal model or not scaling of radiation dominated stage is preferred. Obviously, the needed rescaling of variables and durations acquite in this case the current stage and current time, from where observational information is provided.

The scale factor in descriptive variables for this choice is:

$$\begin{aligned} \bar{\bar{a}}_I(\bar{\bar{t}}) &= \bar{\bar{A}}_I(\bar{\bar{t}}_I - \bar{\bar{t}})^{-Q} & \bar{\bar{t}}_i < \bar{\bar{t}} < t_r \\ \bar{a}_{II}(\bar{t}) &= \bar{A}_{II}(\bar{t})^R & t_r < \bar{t} < t_m \\ a_{III}(t) &= A_{III}(\bar{t} - \bar{t}_{III})^M & t_m < t < \bar{t}_0 \end{aligned} \tag{C.1}$$

where the radiation dominated stage runs on its proper cosmic times among the proper transition times (that we have called also t_r and t_m). Descriptive variables are introduced in inflationary stage and in the current matter dominated stage. The scale factor has continuous and smooth transitions at t_r and t_m and each stage satisfy the proper expansion ratii (3.2-3.4). The parameters of scale factor (C.1) take the expressions:

$$\begin{aligned} \bar{\bar{t}}_I &= t_r\left(1+\frac{Q}{R}\right) \quad , \quad & \bar{t}_{III} &= \left(1-\frac{M}{R}\right)t_m \\ \bar{\bar{A}}_I &= \bar{A}_{II}\left(\frac{Q}{R}\right)^Q (t_r)^{R+Q} \quad , \quad & A_{III} &= \bar{A}_{II}\left(\frac{R}{M}\right)^M {t_m}^{R-M} \end{aligned} \tag{C.2}$$

The descriptive variables take the form:

$$\bar{\bar{t}} = \left(1+\frac{Q}{R}\frac{t-t_r}{t_I-t_r}\right)t_r \quad , \quad \bar{t} = \frac{M}{R}t + \left(1-\frac{M}{R}\right)t_m \tag{C.3}$$

The current time is transformed in this description: $\bar{t}_0 = \frac{M}{R}t_0 + \left(1-\frac{M}{R}\right)t_m$. Also, the beginning of inflationary stage: $\bar{\bar{t}}_i = \left(1+\frac{Q}{R}\frac{t_i-t_r}{t_I-t_r}\right)t_r$. For usual values of temporal dependences, the matter dominated stage suffers a dilatation in these descriptive variables $\Delta\bar{t}|_{MAT} = \frac{M}{R}\,\Delta t|_{MAT}$. For the inflationary stage, the duration is modified as: $\Delta\bar{\bar{t}}\Big|_{INF} = \frac{Q}{R}\frac{t_r}{t_I-t_r}\,\Delta t|_{INF}$. Dilatation takes place also in this stage for $\frac{t_I}{t_r} < \frac{Q}{R} + 1$.

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